MATHEMATICS

SOME THEOREMS ON SPANNING SUBTREES OF A GRAPH *)

BY

E. W. DIJKSTRA

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We consider n points (nodes), all pairs of which are connected by branches of given positive lengths. A tree between these n nodes is a subgraph with one and only one path between any two nodes; a tree consists of $n-1$ branches.

For the sake of simplicity we assume the lengths of the branches to be such that there are no trees of equal length, the length of a tree being defined as the sum of the lengths of its branches; by $T_0, T_1, T_2, \ldots$,we shall denote all possible trees in order of increasing length. Furthermore we assume n to be sufficiently great for our arguments to hold, and state that we shall never make use of the fact that each pair of nodes is connected by exactly one branch. As a matter of fact a number of branches may be missing, provided that the remaining set is large enough for our problems to make sense; we also allow pairs of nodes to be connected by more than one branch.

A closed loop between m branches is defined as a subgraph of m branches with exactly two paths between any two of the m nodes, these two paths being such that they have no branch in common.

Theorem 1. The longest branch of a closed loop never belongs to the shortest tree.

This theorem can be proved by showing that, from a tree containing the longest branch of a closed loop, we can always construct a shorter one. For: remove the branch in question from the tree; the latter now falls apart into two pieces A and B, each node belonging to either A or B. In the closed loop the branch in question connects a node from A to a node from B; when we scan the other path in the loop between these two nodes, we must meet a branch that connects a node from A to a node from B. This branch is shorter than the branch that we removed and is selected to replace the latter. As it connects the two pieces A and B, the resulting set of branches again forms a tree, which is obviously shorter. Theorem 1 is proved in this way.

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Theorem 2. The criterion mentioned in Theorem 1 is sufficient to determine T_0.

We make use of the fact that one loop at most is closed by adding one branch to a graph without loops, and conversely, by removal of a branch of the only loop in a graph, the graph is converted into one without loops. To prove Theorem 2 we consider the following construction. We consider the given branches in an arbitrary order. The first branch is laid down and so is the second one; we continue in this way until a loop is closed. As soon as a loop is closed, we remove the longest branch of the — only! — loop before proceeding to the next branch. When the last branch has been considered, all nodes are connected by the set of branches that has been retained (provided that in the original graph at least one path exists between any two nodes). Furthermore, the remaining graph is one without loops, i.e. it is a tree. Applying Theorem 1, we rejected a number of branches that definitely do not belong to the shortest tree T_0, from the set of all branches. As the remaining graph, however, is a tree, it must be T_0 and Theorem 2 has therefore been proved.

Remark: Every time a new branch is accepted, the "intermediate tree" is shortened, otherwise it remains unaltered.

To my knowledge, two practical methods for finding T_0 have been published, see [1], [2]. They are both special cases of the above process; in both methods the order in which the branches are investigated is chosen in such a way as to facilitate the analysis. We shall now make use of the freedom in this choice of order to prove some theorems. As our construction shows, T_0 is invariant for such modifications of lengths that do not disturb the order of increasing length of the branches. This need no longer hold for T_1, T_2, etc., nevertheless we can prove some theorems concerning them.

Theorem 3. If m branches of the shortest tree T_0 are not contained in the tree T_k, the index k must satisfy the inequality: $k \geqslant 2^m - 1$.

To prove this theorem we carry out the given construction of the shortest tree, starting the investigation with the branches of T_k, i.e. with the tree T_k. We now continue our investigation with an arbitrary non-empty selection from the m branches of T_0. At every such step the new branch will be accepted because it belongs to T_0, and another branch, not belonging to T_0, will be rejected: at every step our intermediate tree is shortened. As the number of non-empty selections from m branches equals $2^m - 1$, we have produced $2^m - 1$ trees, all shorter than T_k and all different from each other. Theorem 3 has therefore been proved. Another way of stating this result is the following:

If k satisfies the inequality $k < 2^m - 1$, than the trees T_k and T_0 have at least $n - m$ branches in common.

From this it follows immediately that both T_1 and T_2 have exactly $n-2$ branches in common with T_0.

If we remove the only branch of T_0 that does not belong to T_1, the latter becomes the shortest tree of the modified graph. The actual construction of T_1 presents some difficulty as we do not know beforehand which branch of T_0 to omit. We therefore omit each of the $n-1$ branches of T_0 in turn, and determine the shortest tree from the remaining graph every time. The tree T_1 must be one of the $n-1$ trees found in this manner, viz. the shortest one. To see what this construction amounts to (once T_0 has been found) we again start the construction of the shortest tree, but postpone the investigation of the $n-1$ branches of T_0, i.e. we determine the shortest tree from the graph that remains when all the branches of T_0 have been removed; let this tree be denoted by $T(0)$. From this point onwards we can proceed in $n-1$ different ways, omitting a different branch of T_0 each time. If we should include such a branch, we would find T_0. From our construction it is clear that the $n-1$ trees, from which T_1 will be selected, each consist of $n-2$ branches of T_0 and $(n-1)-(n-2)=1$ branch of $T(0)$. In other words, the $n-1$ trees are found by omitting a branch from T_0 and by then reconnecting, in the shortest possible way, its two parts with a branch of $T(0)$. Every time our "loss" with respect to T_0 is the difference between a branch of $T(0)$ and one of T_0. The tree T_1 is the one for which this difference is minimal: evidently T_1 need not be invariant for such modifications of lengths that leave the order of increasing length of the branches unaltered.

Theorem 4. Every tree — except the shortest one — contains at most one branch that does not belong to the union of the shorter trees.

We consider an arbitrary tree $T \neq T_0$. Either all its branches are contained in the union of the trees shorter than T, or not. In the first case the conditions stated by Theorem 4 are satisfied, in the second case we must prove that only one of its branches lies outside the union of the shorter trees. To prove this we carry out the construction of the shortest tree, starting the investigation with the branches of T, i.e. with the tree T. From then onwards we investigate the remaining branches in an arbitrary order. We now continue the construction process until a new branch is accepted for the first time. Then one of the branches of T is replaced by another and as a result of this substitution, the tree — being shortened! — is now completely contained in the union of the trees shorter than T. The branch of T that has been replaced at this step was therefore its only branch outside the union and Theorem 4 is proved.

We now consider the trees $T_0, T_1, T_2, T_3, \ldots$, in this order. From Theorem 4 it follows that for every tree one branch or none is added to the union of the trees considered. This process, therefore, defines a unique order for the branches except for those of T_0. This order of the branches

does not coincide with the order of increasing length, although there will be a tendency for the shorter branches to appear in the beginning.

For all our results, the dual statements obtained by interchanging "shorter" and "longer" hold as well. In an analogous manner we can therefore find an order of the branches except for those contained in the longest tree: the longer branches tend to appear in the beginning. Restricting ourselves to branches that belong to neither the shortest, nor the longest tree, we can ask whether the one order is equal to the reverse of the other. An example shows that this need not be the case. Let each pair of the five nodes A, B, C, D, E be connected by a branch, their lengths being given in the following table.

B	1			
C	80	100		
D	20	10	2	
E	24	15	81	77
	A	B	C	D

The shortest tree is (AB, CD, BD, BE); the longest tree is (BC, CE, AC, DE). The remaining branches are AD and AE; the branch AE, however, occurs in the shortest tree but one as well as in the longest tree but one (these trees being (AB, CD, BD, AE) and (BC, CE, DE, AE) respectively).

REFERENCES

1. LOBERMAN, H. and A. WEINBERGER, Formal Procedures for Connecting Terminals with a Minimum Total Wire Length. J. Ass. Comp. Mach. 4, 428–437 (1957).
2. DIJKSTRA, E. W., A Note on Two Problems in Connexion with Graphs. Num. Math. 1 269–271 (1959).